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Critical properties of Dyson's hierarchical model III. The n-vector and Heisenberg models

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Abstract. Exact renormalisation group equations (RGE) are obtained for the d-dimensional n-vector and spin-S Heisenberg versions of Dyson's hierarchical model. In the quantum mechanical case the fixed point and neighbourhood properties of the RGE are shown to be independent of S indicating universality of critical behaviour in spin magnitude. Expansions for critical exponents, analogous to the usual $\epsilon = 4-d$ expansions, are obtained and the RGE solved numerically to obtain relevant scaling indices. The $n \to \infty$ or spherical limit is also discussed and $1/n$ corrections to critical exponents are calculated to first order in $1/n$.

1. Introduction

In two previous papers (Kim and Thompson 1977, 1978, to be referred to as I and II respectively), we investigated the critical behaviour of the one-dimensional hierarchical Ising model (HIM) (Dyson 1969, 1971) utilising a new renormalisation group recursion relation.

In I we derived the renormalisation group equation (RGE) for the one-dimensional HIM with potential falling off as $r^{-(1+\sigma)}$ and investigated the critical behaviour of the model in the range $0<\sigma<1$. Analytic results were obtained in the 'classical' region $0<\sigma<\frac{1}{2}$ and critical exponents were computed to high numerical accuracy in the non-classical region $\frac{1}{2}<\sigma<1$. In addition we obtained $\Delta\sigma = \sigma - \frac{1}{2}$ expansions for critical exponents to third order in $\Delta\sigma$. In II we investigated the borderline one-dimensional HIM, corresponding to $\sigma = 1$, which is known rigorously (Dyson 1971) to undergo a first-order phase transition. By numerical analysis of the RGE we found that the critical point in this case is characterised by essentially singular behaviour.

Our purpose here is to investigate the n-vector and spin-S quantum mechanical Heisenberg versions of the hierarchical model (HM) in arbitrary dimension using the renormalisation group methods developed in I and II.

In § 2 of this paper we derive exact RGE for the n-vector and spin-S Heisenberg hierarchical models in d dimensions. We find that the critical behaviour of the spin-S Heisenberg model is governed by a fixed point of the RGE which is independent of S, and identical with the appropriate fixed point for the classical Heisenberg model ($n=3$, or $S\to\infty$), thus showing explicitly, for the first time to our knowledge, the universality of critical behaviour in spin magnitude for a particular class of quantum mechanical models.

In § 3 we investigate the critical behaviour of the n-vector model by obtaining asymptotic expansions for the scaling index y_E to second order in $\Delta\sigma = \sigma - d/2 > 0$

and numerical values for y_E in the range $d/2<\sigma<d$, for $n=1, 2, 3$ and $d=1$ and 3. (As before all exponents are classical in the range $0<\sigma<d/2$.)

The $n\to\infty$ limit of the n-vector model is discussed in § 4 where it is shown that the equation of state becomes exactly that of the corresponding hierarchical spherical model (McGuire 1973). (In the standard proof of the equivalence of the spherical model and the $n\to\infty$ limit (Kac and Thompson 1971) it is assumed that the interaction is translationally invariant. The hierarchical model is not translationally invariant but all sites are equivalent and the proof can be easily modified to allow for this case.)

In § 5 we discuss the RGE in the $n\to\infty$ limit, obtaining the known result $y_E=d-\sigma$ for $d/2<\sigma<d$ in this limit. An asymptotic expansion for the RGE is also developed and used to obtain the $1/n$ correction to the spherical y_E value.

We conclude in § 6 with a discussion of our results.

2. Renormalisation group equations

For a d-dimensional HM (Blekher and Sinai 1974), a volume V_l with 2^{dl} spins is divided into $r=2^d$ equal subvolumes $V_{l-1,i}$, $i=1,\ldots,r$ each with $2^{d(l-1)}$ spins. The Hamiltonian $\mathscr{H}_l(V_l)$ of the volume V_l is then constructed recursively from $\mathscr{H}_{l-1}(V_{l-1})$ by

$$\mathscr{H}_l(V_l)=\sum_{j=1}^{r}\mathscr{H}_{l-1}(V_{l-1,j})-2^{-(d+\sigma)l}\Big(\sum_{i\in V_l}\boldsymbol{S}_{0,i}\Big)^2 \tag{2.1}$$

where for the n-vector model $\boldsymbol{S}_{0,i}$ is the n-dimensional classical spin at position i with magnitude $\|\boldsymbol{S}_{0,i}\|$ equal to unity. The starting point of (2.1) is

$$\mathscr{H}_0(V_{0,i})=-\boldsymbol{H}\cdot\boldsymbol{S}_{0,i} \tag{2.2}$$

where $\boldsymbol{H}$ is the n-dimensional magnetic field vector.

As in I, we use the identity

$$\exp(ay^2)=\pi^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-x^2+2a^{1/2}\boldsymbol{x}\cdot\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \tag{2.3}$$

where

$$x=\|\boldsymbol{x}\|,\qquad a=\beta 2^{-(d+\sigma)l}\qquad\text{and}\qquad \boldsymbol{y}=\sum_{i\in V_l}\boldsymbol{S}_{0,i},$$

to decouple the interaction term in (2.1) and thereby obtain the recursion relation for the partition function:

$$Q_l(\beta,\boldsymbol{H})=\int_{\|\boldsymbol{S}_{0,i}\|=1}\cdots\int\exp(-\beta\mathscr{H}_l(V_l))\prod\mathrm{d}\boldsymbol{S}_{0,i}$$

$$=\pi^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-x^2)(Q_{l-1}(\beta,\boldsymbol{H}+v_l\boldsymbol{x}))^r\,\mathrm{d}\boldsymbol{x} \tag{2.4}$$

where $v_l=2\beta^{-1/2}2^{-(d+\sigma)l/2}$. Defining

$$\tilde{P}_l(\boldsymbol{y})=Q_l(\beta,v_{l+1}\boldsymbol{y}), \tag{2.5}$$

the RGE can be expressed as

$$\tilde{P}_l(2^{(d+\sigma)/2}\boldsymbol{y}) = \pi^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp[-(\boldsymbol{x}-\boldsymbol{y})^2](\tilde{P}_{l-1}(\boldsymbol{x}))^r\,\mathrm{d}\boldsymbol{x}. \tag{2.6}$$

Furthermore, due to the spherical symmetry in $\boldsymbol{H}$ of $Q_l(\beta, \boldsymbol{H})$, $\tilde{P}_l(\boldsymbol{x})$ depends only on $x = \|\boldsymbol{x}\|$. The angular integration in (2.6) can then be performed easily to give

$$\tilde{P}_l(2^{(d+\sigma)/2}y) = R_n(x, y)\circ(\tilde{P}_{l-1}(x))^r \tag{2.7}$$

where the integral operator $R_n(x, y)$ is defined as

$$R_n(x, y)\circ f(x) = 2\int_0^{\infty}\exp[-(x^2+y^2)]y^{1-\frac{1}{2}n}x^{n/2}I_{\frac{1}{2}n-1}(2xy)f(x)\,\mathrm{d}x, \tag{2.8}$$

with $r = 2^d$ and $I_\nu(z)$ the modified Bessel function of the first kind of order ν.

Equations (2.7) and (2.8) are the desired RGE for the n-vector HM. We note here that for $n = 1$ and $n = 3$ (2.7) reduces to

$$\tilde{P}_l(2^{(d+\sigma)/2}y) = \pi^{-1/2}\int_{-\infty}^{\infty}\exp[-(x-y)^2](\tilde{P}_{l-1}(x))^r\,\mathrm{d}x \tag{2.9}$$

and

$$\tilde{P}_l(2^{(d+\sigma)/2}y) = \pi^{-1/2}\int_{-\infty}^{\infty}\exp[-(x-y)^2]\frac{x}{y}(\tilde{P}_{l-1}(x))^r\,\mathrm{d}x \tag{2.10}$$

respectively, if we use the fact that $\tilde{P}_l(x)$ is an even function of x. (The definition of $\tilde{P}_l(x)$ is slightly different from that considered in I.)

For the spin-S Heisenberg HM, we write the Hamiltonian in the form

$$\mathcal{H}_l(V_l) = \sum_{j=1}^{r}\mathcal{H}_{l-1}(V_{l-1,j}) - S^{-2}2^{-(d+\sigma)l}\left[\left(\sum_{i\in V_l}\hat{\boldsymbol{S}}_{0,i}\right)^2 + \tfrac{1}{4}\right] \tag{2.11}$$

where the notation is the same as in (2.1) except that now $\hat{\boldsymbol{S}}_{0,i}$ are quantum mechanical spins of magnitude S with $(\hat{\boldsymbol{S}}_{0,i})^2 = S(S+1)$. The constant term is added for later convenience. The single spin Hamiltonian is

$$\mathcal{H}_0(V_{0,i}) = -\frac{H}{S}(\hat{S}_{0,i})_z \tag{2.12}$$

where H is the magnetic field in the z direction.

To derive the RGE for this system, we restrict our discussion for the moment to the $d = 1$ case for notational simplicity. Written explicitly, the Hamiltonian for the N-level 2^N spin system is

$$\mathcal{H}_N = -S^{-2}\sum_{p=1}^{N}2^{-(1+\sigma)p}\sum_{r=1}^{2^{N-p}}[(\hat{\boldsymbol{S}}_{p,r})^2 + \tfrac{1}{4}] - H/S(\hat{S}_{N,1})_z \tag{2.13}$$

where

$$\hat{\boldsymbol{S}}_{p,r} = \hat{\boldsymbol{S}}_{p-1,2r-1} + \hat{\boldsymbol{S}}_{p-1,2r} = \sum_i \hat{\boldsymbol{S}}_{0,i}$$

for $(r-1)2^p + 1 \leqslant i \leqslant r2^p$, $r = 1, \ldots, 2^{N-p}$, $p = 1, \ldots, N$. As shown by Dyson (1969), the partition function for (2.13) may be written as

$$Q_N(\beta, H)=\sum_{\{j_{p,r}\}}\sum_{m=-j_{N,1}}^{j_{N,1}} \exp[-\beta E_N(\{j_{p,r}\})+\beta Hm/S] \tag{2.14}$$

where the eigenvalue $E_N(\{j_{p,r}\})$ of $\mathscr{H}_N$ in the diagonal representation is

$$E_N(\{j_{p,r}\})=-S^{-2}\sum_{p=1}^{N} 2^{-(1+\sigma)p}\sum_{r=1}^{2N-p}(j_{p,r}+\tfrac{1}{2})^2. \tag{2.15}$$

The first sum in (2.14) is over all integers satisfying the conditions;

$$\begin{aligned}&0\leqslant j_{1,r}\leqslant 2S; \qquad r=1,\dots,2^{N-1}\\ &|j_{p-1,2r-1}-j_{p-1,2r}|\leqslant j_{p,r}\leqslant j_{p-1,2r-1}+j_{p-1,2r};\\ &p=2,\dots,N, \quad r=1,\dots,2^{N-p}.\end{aligned} \tag{2.16}$$

If we consider two such systems of 2^N-spins coupled together by the $(N+1)$th level interaction, thus forming the $N+1$ level system, the partition function of the latter becomes

$$\begin{aligned}Q_{N+1}(\beta, H)=&\sum_{\{j_{p,r}\}}\sum_{\{j'_{p,r}\}}\sum_{j=|j_{N,1}-j'_{N,1}|}^{j_{N,1}+j'_{N,1}}\sum_{m=-j}^{j}\\ &\times\exp[-\beta E_N(\{j_{p,r}\})-\beta E_N(\{j'_{p,r}\})+\beta S^{-2}2^{-(1+\sigma)(N+1)}(j+\tfrac{1}{2})^2+\beta Hm/S]\end{aligned} \tag{2.17}$$

where $\{j_{p,r}\}$ and $\{j'_{p,r}\}$ denote the eigenstates of $\{S^2_{p,r}\}$ for the two N-level systems respectively. Now let us consider the following quantity:

$$\begin{aligned}A&\equiv\sum_{j=|x-y|}^{x+y}\sum_{m=-j}^{j}\exp[(\beta v/2S)^2(j+\tfrac{1}{2})^2+\beta Hm/S]\\ &=\sum_{j=|x-y|}^{x+y}\exp[(\beta v/2S)^2(j+\tfrac{1}{2})^2]\frac{\sinh[(j+\tfrac{1}{2})\beta H/S]}{\sinh(\beta H/2S)}.\end{aligned} \tag{2.18}$$

Using the identity

$$\exp(b^2/4)\sinh a=\pi^{-1/2}\int_{-\infty}^{\infty}\exp(-z^2)\sinh(a+bz)\,\mathrm{d}z \tag{2.19}$$

which is the analogue of (2.3), in equation (2.18) with $b=\beta v(j+\frac{1}{2})/S$ and $a=(j+\frac{1}{2})\beta H/S$, A may be expressed as

$$A=\pi^{-1/2}\int_{-\infty}^{\infty}\exp(-z^2)\sum_{j=|x-y|}^{x+y}\frac{\sinh[(j+\tfrac{1}{2})\beta(H+vz)/S]}{\sinh(\beta H/2S)}\,\mathrm{d}z. \tag{2.20}$$

If we perform the sum in equation (2.20) explicitly, we then get

$$\begin{aligned}A&=\pi^{-1/2}\int_{-\infty}^{\infty}\exp(-z^2)\frac{\sinh[(x+\tfrac{1}{2})\beta H'/S)\sinh[(y+\tfrac{1}{2})\beta H'/S]}{\sinh(\beta H/2S)\sinh(\beta H'/2S)}\,\mathrm{d}z\\ &=\pi^{-1/2}\int_{-\infty}^{\infty}\exp(-z^2)\frac{\sinh(\beta H'/2S)}{\sinh(\beta H/2S)}\left(\sum_{m=-x}^{x}\exp(\beta H'm/S)\right)\\ &\quad\times\left(\sum_{m=-y}^{y}\exp(\beta H'm/S)\right)\mathrm{d}z\end{aligned} \tag{2.21}$$

where $H' = H + vz$. Now substituting (2.21) into (2.17) with $x = j_{N,1}$, $y = j'_{N,1}$, $v = 2\beta^{-1/2}2^{-(1+\sigma)(N+1)/2}$, the sums over $\{j_{p,r}\}$ and $\{j'_{p,r}\}$ are decoupled and each gives the partition function of the N-level system with a modified magnetic field $H' = H + vz$. Hence (2.17) now becomes

$$Q_{N+1}(\beta, H) = \pi^{-1/2} \int_{-\infty}^{\infty} \exp(-z^2) \frac{\sinh[\beta(H+vz)/2S]}{\sinh(\beta H/2S)} (Q_N(\beta, H+vz))^2 \, dz. \tag{2.22}$$

This is the Heisenberg HM analogue of equation (2.4) for $d = 1$. This extension for general d is straightforward and the result is

$$Q_{N+1}(\beta, H) = \pi^{-1/2} \int_{-\infty}^{\infty} \exp(-z^2) \frac{\sinh[\beta(H+v_{N+1}z)/2S]}{\sinh(\beta H/2S)} (Q_N(\beta, H+v_{N+1}z))^r \, dz \tag{2.23}$$

where $v_N = 2\beta^{-1/2}2^{-(d+\sigma)N/2}$ and $r = 2^d$.

Proceeding as before we define $\tilde{P}_l(y)$ by

$$\tilde{P}_l(y) = Q_l(\beta, v_{l+1}y) \tag{2.24}$$

to obtain the RGE for the spin-S Heisenberg HM:

$$\tilde{P}_l(2^{(d+\sigma)/2}y) = \pi^{-1/2} \int_{-\infty}^{\infty} \exp[-(x-y)^2] K_l(x, y) (\tilde{P}_{l-1}(x))^r \, dx \tag{2.25}$$

where the kernel $K_l(x, y)$ is defined by

$$K_l(x, y) = \frac{\sinh(\beta v_l x/2S)}{\sinh(\beta v_l y/2S)}. \tag{2.26}$$

We note that due to (2.26) the renormalisation group transformation in (2.25) is both spin and l dependent.

When $S \to \infty$, $K_l(x, y)$ becomes simply x/y so that (2.25) reduces to the classical Heisenberg ($n = 3$) result (2.10) as expected. At criticality ($\beta = \beta_c$), a normalised $\tilde{P}_l(x)$, e.g. $\tilde{P}_l(x)/\tilde{P}_l(0)$, approaches a fixed point as $l \to \infty$ and the critical properties of the model are determined from a neighbourhood analysis of the fixed point. However, since $K_l(x, y) \to x/y$ as $l \to \infty$ for all S, the fixed point of (2.25) and its neighbourhood properties are *independent of* S. It follows that for this model we have universality of critical behaviour in spin magnitude S. That is, the critical exponents for the spin-S Heisenberg HM (2.11) and (2.12) are independent of S and are identical to those of the classical Heisenberg HM.

3. Critical exponents of the n-vector HM

As in the $n = d = 1$ case of I, the Gaussian fixed point

$$\tilde{P}^*(x) = \exp[2^{-d}(1-2^{-\sigma})x^2 - \tfrac{1}{2}\sigma n(2^d-1)^{-1} \ln 2] \tag{3.1}$$

of the n-vector RGE (2.7) is thermodynamically stable for the range of σ, $0 < \sigma < d/2$. Thus in this range of σ, the critical exponents assume their classical values $\gamma = 1$, $\nu = 1/\sigma$ etc (see also Guttman *et al* 1977). From now on, we restrict our attention to the non-classical region, $d/2 \leqslant \sigma < d$. The quantities of interest here are the two relevant scaling exponents y_E and y_H (Wegner 1976) which characterise the homogeneity of the singular part of the free energy;

$$F_s(t, H) = 2^{-dl} F_s(2^{y_E l} t, 2^{y_H l} H). \tag{3.2}$$

Due to the nature of contruction of RGE (2.7), y_H is given simply by

$$y_H = \tfrac{1}{2}(d + \sigma) \tag{3.3}$$

and it follows that $\delta = (d+\sigma)/(d-\sigma)$ and $\eta = 2-\sigma$. The main object of concern in this section is the calculation of y_E from which other critical exponents are obtained; for example, $\gamma = \sigma/y_E$ and $\nu = 1/y_E$.

In I, we obtained an asymptotic expansion of y_E in powers of $\Delta\sigma = \sigma - d/2$ to third order in $\Delta\sigma$ for $n = d = 1$. Extension of the expansion to the more general case is straightforward and here we only outline the derivation. First, we take out the Gaussian part (3.1) from $\tilde{P}_l(x)$ by defining $h_l(x)$ as

$$\tilde{P}_l(x) = \tilde{P}^*(x)(h_l(2^{-\sigma/2}x))^{1/r} \tag{3.4}$$

where $r = 2^d$ as before and $\tilde{P}^*(x)$ is given in (3.1). The RGE (2.7) in terms of $h_l(x)$ then becomes

$$h_{l+1}(2^{(d-\sigma)/2}y) = (R_n(x, y) \circ h_l(x))^r \tag{3.5}$$

where the operator $R_n(x, y)$ is defined by (2.8). The eigenfunctions of $R_n(x, y)$ are the generalised Laguerre polynomials $L_k^\alpha(x)$. This follows from the identity

$$R_n(x, y) \circ L_k^{\frac{1}{2}n-1}(ax^2) = (1-a)^k L_k^{\frac{1}{2}n-1}(a(1-a)^{-1}y^2) \tag{3.6}$$

which can be easily obtained from equation (5), p 43 of Erdélyi *et al* (1954). Here we choose the constant a to be

$$a = 1 - 2^{-(d-\sigma)} \tag{3.7}$$

to match the scale factor in (3.5).

By expanding $h_l(x)$ as

$$h_l(x) = A_0^{(l)}\left(1 + \sum_{k=1}^{\infty} A_k^{(l)} 2^{(d-\sigma)k} L_k^{\frac{1}{2}n-1}(ax^2)\right) \tag{3.8}$$

the RGE (3.5) can be expressed as

$$A_0^{(l+1)}\left(1 + \sum_{k=1}^{\infty} A_k^{(l+1)} 2^{(d-\sigma)k} L_k^{\frac{1}{2}n-1}(ay^2)\right) = (A^{(l)})^r \left(1 + \sum_{k=1}^{\infty} A_k^{(l)} L_k^{\frac{1}{2}n-1}(ay^2)\right)^r. \tag{3.9}$$

Expanding the right-hand side of (3.9) and utilising, iteratively, the identity (Erdélyi *et al* 1953):

$$L_1^\alpha L_k^\alpha = (k+1)L_{k+1}^\alpha - 2kL_k^\alpha + (k+\alpha)L_{k-1}^\alpha, \tag{3.10}$$

products of the $L_k^{\frac{1}{2}n-1}$ can be expressed as linear combinations of the $L_k^{\frac{1}{2}n-1}$ so that finally the RGE (3.5) can be expressed algebraically in terms of the $A_k^{(l)}$. We then find the non-Gaussian fixed point of (3.9) for which A_2^* is of order $\Delta\sigma$ and by linearising (3.9) about that fixed point, we calculate the largest eigenvalue Λ_1 of the linearised transformation relating the $A_k^{(l)}$ to the $A_k^{(l+1)}$, to second order in $\Delta\sigma$. The method follows closely that of I so we simply state the final result here for $y_E = \ln \Lambda_1/\ln 2$, namely,

$$y_E = \frac{d}{2}\left[1 + 2\left(\frac{4-n}{n+8}\right)\frac{\Delta\sigma}{d} - \frac{8(n+2)(7n+20)}{(n+8)^3}\Psi(d)\left(\frac{\Delta\sigma}{d}\right)^2 + \ldots\right] \qquad (3.11)$$

where

$$\Psi(d) = (r+1+4r^{1/2})(r-1)^{-1}\ln r \qquad \text{(HM)} \qquad (3.12)$$

and $r = 2^d$. This is to be compared with the $\Delta\sigma$ expansion for the power law potential n-vector model obtained by Fisher *et al* (1972). Their result for y_E is also given by (3.11) (for $\sigma < 2$) except that $\Psi(d)$ is given by

$$\Psi(d) = d(\psi(1) - 2\psi(d/4) + \psi(d/2)) \qquad \text{(power law model)} \qquad (3.13)$$

where $\psi(z) = d \ln \Gamma(z)/\mathrm{d}z$, and $\Gamma(z)$ is the gamma function. It is interesting to note that the n dependence of the critical exponents is exactly the same for the two models, at least to second order in $\Delta\sigma$.

For the marginal case $\sigma = d/2$ the analysis given in *I* can be repeated for general n to show that the susceptibility behaves as

$$\chi \sim t^{-1}(\ln t^{-1})^{(n+2)/(n+8)} \qquad \text{as } t = 1 - \frac{\beta}{\beta_c} \to 0+. \qquad (3.14)$$

This is exactly the same behaviour found by Fisher *et al* (1972) for the power law model.

The RGE (2.7) is equally well suited for numerical calculation of y_E for any n, d and σ since it can be readily computerised if, as in II, we terminate the infinite power series for $\tilde{P}_l(x)$ at $2M$th order for sufficiently large M and use the identity

$$R_n(x, y) \circ x^{2k} = k! L_k^{\frac{1}{2}n-1}(-y^2). \qquad (3.15)$$

By iterating (2.7), starting from

$$\tilde{P}_0(x) = Q_0(\beta, v_1 x) = \Gamma(n/2)(\beta v_1 x/2)^{-\frac{1}{2}n+1} I_{\frac{1}{2}n-1}(\beta v_1 x) \qquad (3.16)$$

where $v_1 = 2\beta^{-1/2}2^{-(d+\sigma)/2}$, we calculated the susceptibility $\chi(\beta)$ for given n, d and σ directly using the relation (2.5) for various β just below the critical temperature β_c. Then by fitting χ to the usual power law form for $1 - \beta/\beta_c$ as small as 10^{-10} we could obtain reasonably accurate values of γ for most cases considered. The values of γ obtained in this way for $n = 1$, 2 and 3 and for several values of σ in the range $\frac{1}{2} < \sigma/d < 1$ are shown in tables 1 and 2 for $d = 1$ and 3 respectively. The dimensional effect of d on γ, as we see from the tables, is very weak. This in fact could have been anticipated from the $\Delta\sigma$ expansion (3.11). In (3.11), if y_E/d is considered as a function of n, σ/d and d, the d dependence to the second order in $\Delta\sigma$ comes only through the factor $\Psi(d)$ which is a very slowly varying function for $d \lesssim 3$. For example, the values of $\Psi(d)$ for $d = 1$, 2 and 3 are 6·0005, 6·0073 and 6·0345 respectively.

The values of y_E/d as a function of σ/d at $d = 1$ are plotted in figure 1 for $n = 1$, 2 and 3, together with the $n = \infty$ results of the next sections. Similar curves for $d = 3$ are

Table 1. The values of γ for $n = 1, 2$ and 3 at $d = 1$ for several values of σ. The values for $n = 1$ are reproduced from I.

σ/d	$n = 1$	$n = 2$	$n = 3$
0·55	1·07	1·09	1·11
0·60	1·16	1·21	1·25
0·65	1·262	1·356	1·440
0·70	1·381	1·561	1·728
0·75	1·530	1·873	2·202
0·80	1·731	2·428	3·072
0·85	2·030	3·658	4·747
0·90	2·571	7·01	8·16
0·95	4·037	17·4	18·2

Table 2. The value of γ for $n = 1, 2$ and 3 at $d = 3$ for several values of σ.

σ/d	$n = 1$	$n = 2$	$n = 3$
0·55	1·07	1·09	1·11
0·60	1·16	1·21	1·25
0·65	1·263	1·357	1·442
0·70	1·383	1·564	1·732
0·75	1·535	1·882	2·210
0·80	1·741	2·45	3·08
0·85	2·056	3·68	—
0·90	2·64	—	—
0·95	4·3	—	—

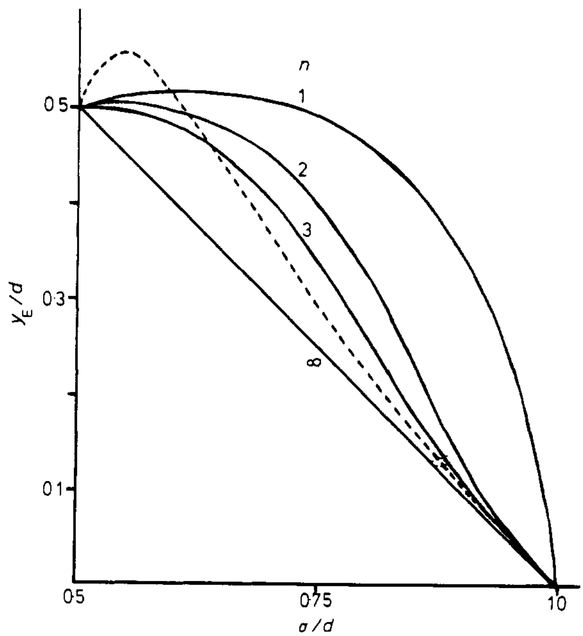

Figure 1. The behaviour of y_E for $n = 1$, 2, 3 and ∞ at $d = 1$. The broken curve is the approximate value of y_E for $n = 3$ obtained by including the $1/n$ connection term (equation (5.21)).

almost indistinguishable from those of figure 1 on this scale. We also show in tables 3 and 4 the dependence of the critical temperature β_c on n and σ for $d = 1$ and 3 respectively.

Table 3. β_c/n, the normalised critical temperatures for $n = 1$, 2 and 3 at $d = 1$ together with the exact values for $n = \infty$. The values for $n = 1$ are reproduced from I.

σ/d	$n = 1$	$n = 2$	$n = 3$	$n = \infty$
0·55	0·39691	0·40559	0·41113	0·43298
0·60	0·47113	0·48574	0·49526	0·53245
0·65	0·55754	0·58189	0·59818	0·66055
0·70	0·65901	0·69968	0·72787	0·83157
0·75	0·77969	0·84884	0·89914	1·07130
0·80	0·92602	1·0488	1·1438	1·43126
0·85	1·1091	1·3486	1·5422	2·03166
0·90	1·3514	1·9223	2·3398	3·23318
0·95	1·7185	3·702	4·74	6·83920

Table 4. β_c/n, the normalised critical temperatures for $n = 1$, 2 and 3 at $d = 3$ together with the exact values for $n = \infty$ (equation (4.29)).

σ/d	$n = 1$	$n = 2$	$n = 3$	$n = \infty$
0·55	1·4335	1·4570	1·4726	1·5394
0·60	1·7342	1·7756	1·8037	1·9230
0·65	2·0920	2·1639	2·2144	2·4232
0·70	2·5204	2·6459	2·7377	3·0986
0·75	3·0389	3·2624	3·4345	4·0543
0·80	3·6770	4·0943	4·4347	5·5008
0·85	4·4844	5·3441	6·064	7·9291
0·90	5·5632	7·714	—	12·8118
0·95	7·2166	—	—	27·5128

4. Equation of state in the $n \to \infty$ limit

For translationally invariant interactions, it is well known that the n-vector model in the $n \to \infty$ limit is equivalent to the spherical model (Kac and Thompson 1971). For the HM all sites are equivalent but the interaction is not translationally invariant. With only slight modifications, however, the proof goes through in the weaker case of equivalent sites. In this section we solve the n-vector HM exactly in the $n \to \infty$ limit and show that the resulting equation of state for general d is in fact identical to that of the spherical HM. The methods developed here are of interest in their own right and will be used in the following section to develop $1/n$ expansions.

In order to investigate the $n \to \infty$ limit, we take the spin magnitude of each $\boldsymbol{S}_{0,i}$ in (2.1) and (2.2) to be

$$\|\boldsymbol{S}_{0,i}\| = n^{1/2} \tag{4.1}$$

instead of unity and let $\|\boldsymbol{H}\|$ be of order of $n^{1/2}$. Thus if we let

$$\|\boldsymbol{H}\| = H = n^{1/2}\beta^{-1}h \tag{4.2}$$

the single spin partition function then becomes

$$Q_0(\beta, H) = A_n^{-1} \int_{\|x\|=n^{1/2}} \cdots \int \exp(\beta \boldsymbol{H} . \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

$$= 2(\pi n)^{(n-1)/2}\left(\frac{A_n}{\Gamma(\frac{1}{2}(n-1))}\right)^{-1} \int_0^{\pi} \exp(nh \cos\theta)(\sin\theta)^{n-2} \, \mathrm{d}\theta \tag{4.3}$$

where the normalisation factor A_n is the surface area of the n-dimensional sphere of radius $n^{1/2}$:

$$A_n = 2\pi^{n/2} n^{(n-1)/2}/\Gamma(n/2). \tag{4.4}$$

Now we define the 'free energy' of the l-level, r^l-spin n-vector HM by

$$f_l^{(n)}(h) = n^{-1} r^{-l} \ln Q_l(\beta, n^{1/2}\beta^{-1}h) \tag{4.5}$$

where $r = 2^d$ and the l-level partition function $Q_l(\beta, H)$ satisfies the recursion relation (2.4) with initial value (4.3). In terms of $f_l^{(n)}$, (2.4) reduces to

$$\exp(nr^l f_l^{(n)}(y))$$

$$= 2\pi^{-1/2}(a_l n)^{n/2}[\Gamma(\tfrac{1}{2}(n-1))]^{-1}$$

$$\times \int_0^{\infty} \int_{-1}^{1} \exp[-a_l n(x^2 - 2xyt + y^2) + nr^l f_{l-1}^{(n)}(x)] x^{n-1}(1-t^2)^{(n-3)/2} \, \mathrm{d}x \, \mathrm{d}t \tag{4.6}$$

where

$$a_l = (\beta v_l)^{-2} = (4\beta)^{-1} 2^{(d+\sigma)l}. \tag{4.7}$$

In the $n \to \infty$ limit, (4.6) and (4.3) can be evaluated immediately by Laplace's method. Thus if

$$f_l(h) = \lim_{n\to\infty} f_l^{(n)}(h) \tag{4.8}$$

for $l = 0, 1, 2, \ldots$, we have

$$f_l(y) = \max_{0 \leqslant x < \infty} \max_{|t| \leqslant 1} [-r^{-l} a_l(x^2 + y^2 - 2xyt) + f_{l-1}(x) + r^{-l} \ln x$$

$$+ \tfrac{1}{2} r^{-l} \ln(1-t^2) + \tfrac{1}{2} r^{-l} + \tfrac{1}{2} r^{-l} \ln(2a_l)] \tag{4.9}$$

and

$$f_0(y) = \max_{|t| \leqslant 1} [yt + \tfrac{1}{2} \ln(1-t^2)] \tag{4.10}$$

from (4.6) and (4.3) respectively.

It seems very difficult to obtain a closed form expression for $f_l(h)$ from (4.9) and (4.10). However, if we take several transformations of (4.9), we can arrive at a simple recursion relation which can be solved. First we take the derivatives of (4.9) and (4.10) with respect to y to obtain a recursion relation for the magnetisation per spin in field h,

$$m_l(h) = \frac{\mathrm{d}}{\mathrm{d}h} f_l(h). \tag{4.11}$$

Defining $g_l(h)$ by

$$m_l(h)=hg_l(h^2). \tag{4.12}$$

Equation (4.9) can then be brought, after some algebra, into the form

$$g_l(y)=g_{l-1}(x)(1-\tfrac{1}{2}r^l a_l^{-1}g_{l-1}(x))^{-1} \tag{4.13}$$

where x is a function of y determined by

$$y=x(1-\tfrac{1}{2}r^l a_l^{-1}g_{l-1}(x))^2-\tfrac{1}{2}a_l^{-1}(1-\tfrac{1}{2}r^l a_l^{-1}g_{l-1}(x)). \tag{4.14}$$

Also, in terms of $g_0(y)$, equation (4.10) becomes

$$g_0(y)=(2y)^{-1}[(1+4y)^{1/2}-1]. \tag{4.15}$$

Our next step, which is crucial for the solution, is to express (4.13) and (4.14) in terms of the inverse function of $g_0(x)$. That is, if

$$g_l^{-1}(g_l(x))=x \tag{4.16}$$

and if we further define

$$G_l(y)=y^2 g_l^{-1}(y) \tag{4.17}$$

we finally arrive at the functional recursion equation

$$G_l(y)=G_{l-1}(y(1+\alpha_l y)^{-1})-\tfrac{1}{2}a_l^{-1}y^2(1+\alpha_l y)^{-1} \tag{4.18}$$

where

$$\alpha_l=\tfrac{1}{2}r^l a_l^{-1}=2\beta 2^{-\sigma l}, \tag{4.19}$$

and

$$G_0(y)=1-y. \tag{4.20}$$

The recursion relation (4.18) is now simple enough to obtain an explicit expression for $G_l(y)$. The result is

$$G_L(y)=1-y(1+A_{1,L}y)^{-1}-\tfrac{1}{2}y^2\sum_{l=1}^{L}a_l^{-1}(1+A_{l,L}y)^{-1}(1+A_{l+1,L}y)^{-1} \tag{4.21}$$

where

$$A_{l,L}=\sum_{k=l}^{L}\alpha_k=2\beta(1-2^{-\sigma})^{-1}2^{-\sigma l}(1-2^{-\sigma(L-l+1)}). \tag{4.22}$$

Due to definitions (4.12), (4.16) and (4.17),

$$G_L(m_L(h)/h)=(m_L(h))^2. \tag{4.23}$$

Hence, if we take the thermodynamic limit $L\to\infty$ in (4.21) and (4.23) and if

$$m(h)=\lim_{L\to\infty}m_L(h),$$

we then get the equation of state

$$m^2=F(h/m) \tag{4.24}$$

where

$$F(x)=G_\infty(x^{-1})=1-(1-2^{-d})\sum_{l=0}^{\infty}2^{-dl}[x+2\beta(2^\sigma-1)^{-1}2^{-\sigma l}]^{-1}. \tag{4.25}$$

On the other hand, for the spherical hierarchical model, after correcting some misprints in McGuire (1973), and generalising it for general d, we find (D Kim, unpublished)

$$\beta A = \tfrac{1}{2}(1-2^{-d}) \sum_{l=0}^{\infty} 2^{-dl}(s+2^{-\sigma p})^{-1} + \tfrac{1}{4}h^2\beta^{-1}A^{-1}s^{-2} \tag{4.26}$$

for the spherical constraint equation and

$$m = \tfrac{1}{2}(\beta A s)^{-1} h \tag{4.27}$$

for the magnetisation, where $A = (2^{\sigma}-1)^{-1}$, and h is the reduced magnetic field. When we eliminate s between (4.26) and (4.27), we get identical results to (4.24) and (4.25).

Most critical properties for the $n \to \infty$ HM follow from (4.24). For example, for $\beta > \beta_c$ where $m > 0$ as $h \to 0$, we find

$$m^2 = F(0) = 1 - \beta_c/\beta \tag{4.28}$$

where

$$\beta_c = \tfrac{1}{2}(2^d-1)(2^{\sigma}-1)(2^d-2^{\sigma})^{-1} \tag{4.29}$$

and since

$$F(x) - F(0) \sim x^{(d-\sigma)/\sigma} \qquad \text{for } d/2 < \sigma < d,$$

(McGuire 1973), we obtain

$$y_E = d - \sigma. \tag{4.30}$$

5. Renormalisation group equations in the $n \to \infty$ limit and $1/n$ expansions

In this section, we investigate the critical properties of the n-vector HM for large n in the context of the renormalisation group. More specifically, we derive here the $1/n$ connection to the $n \to \infty$ limit result of y_E, (4.30). For this purpose, we need to put the $n \to \infty$ limit result in a RGE form. The line of approach is similar to the one given in § 4.

Starting from the RGE (3.5) and defining $q_l(x)$ by

$$h_l(x) = \exp(nq_l(n^{-1/2}x)) \tag{5.1}$$

in an analogous manner to (4.5), equation (3.5) may be written as

$$\begin{aligned} \exp(n2^{-d}q_{l+1}(2^{(d-\sigma)/2}y)) &= 2\pi^{-1/2}n^{n/2}[\Gamma\tfrac{1}{2}(n-1)]^{-1} \\ &\quad \times \int_0^{\infty}\int_{-1}^{1} \exp[-n(x^2-2xyt+y^2)+nq_l(x)]x^{n-1}(1-t^2)^{(n-3)/2}\,dx\,dt. \end{aligned} \tag{5.2}$$

Now if we define $\tilde{g}_l(x)$ by

$$\frac{dq_l(x)}{dx} = 2x\tilde{g}_l(x^2) \tag{5.3}$$

as in (4.11) and (4.12), and follow similar steps to those given in § 4, (5.2) becomes, in the limit $n \to \infty$,

$$\tilde{g}_{l+1}(2^{d-\sigma}y) = 2^{\sigma}\tilde{g}_l(x)(1-\tilde{g}_l(x))^{-1} \tag{5.4}$$

where x and y are related by

$$y = x(1-\tilde{g}_l(x))^2 - \tfrac{1}{2}(1-\tilde{g}_l(x)). \tag{5.5}$$

Again, as in § 4, we represent (5.4) and (5.5) in terms of the inverse function $\tilde{g}_l^{-1}(x)$ of $\tilde{g}_l(x)$ as

$$2^{d-\sigma}y = \tilde{g}_{l+1}^{-1}(2^{\sigma}u(1-u)^{-1}) = 2^{d-\sigma}[\tilde{g}_l^{-1}(u)(1-u)^2 - \tfrac{1}{2}(1-u)] \tag{5.6}$$

and we further define $\tilde{G}_l(y)$ by

$$\tilde{G}_l(y) = (1-y)^{-2}\tilde{g}_l^{-1}((2^{\sigma}-1)y(1-y)^{-1}). \tag{5.7}$$

We then finally arrive at the transformed RGE

$$\tilde{G}_{l+1}(2^{\sigma}y) = 2^{d-\sigma}[\tilde{G}_l(y) - \tfrac{1}{2}(1-y)^{-1}(1-2^{\sigma}y)^{-1}]. \tag{5.8}$$

It should be mentioned here that for the Gaussian fixed point (3.1) which corresponds to $\tilde{g}^* = 0$ in (5.4), the corresponding $\tilde{G}^*$ is not defined. Thus (5.8), is only applicable to the non-Gaussian fixed point which is of course our primary concern here.

If the fixed point $\tilde{G}^*$ of (5.8) is written as

$$\tilde{G}^*(y) = \sum_{k=0}^{\infty} A_k^* y^k, \tag{5.9}$$

(assuming analyticity of $\tilde{G}^*(y)$ at $y = 0$) the A^* are simply given by

$$A_k^* = \tfrac{1}{2}r(s-1)^{-1}(s^{k+1}-1)(r-s^{k+1})^{-1} \tag{5.10}$$

where $r = 2^d$ and $s = 2^{\sigma}$. Furthermore, if we let

$$\tilde{G}_l(y) = \tilde{G}^*(y) + \sum_{k=0}^{\infty} a_k^{(l)} y^k \tag{5.11}$$

we immediately see that

$$a_k^{(l+1)} = 2^{d-\sigma-\sigma k} a_k^{(l)}. \tag{5.12}$$

In other words, y^k, $k = 0, 1, 2, \ldots$, are the 'eigen-operators' of the RGE (5.8) and in the range $\tfrac{1}{2}d < \sigma < d$, only one, corresponding to $k = 0$, is relevant with a scaling exponent y_{E} given by (4.30), in accordance with the results of the previous section.

To obtain the $1/n$ correction to y_{E}, we use the asymptotic form

$$\int_a^b \exp(nf(x))\,\mathrm{d}x = (2\pi)^{1/2}(-nf''(x_0))^{-1/2}\exp(nf(x_0))[1+\mathrm{O}(n^{-1})] \tag{5.13}$$

where $f'(x_0) = 0$ and $a < x_0 < b$, in (5.2) twice and simply retrace the steps from (5.2) to (5.12) taking careful account of the $1/n$ order terms. When the calculations are carried through to the stage of (5.8), the RGE becomes

$$\tilde{G}_{l+1}(2^{\sigma}y) = 2^{d-\sigma}[\tilde{G}_l(y) - \tfrac{1}{2}(1-y)^{-1}(1-2^{\sigma}y)^{-1}] + n^{-1}Z\left(y, \tilde{G}_l(y), \frac{\mathrm{d}\tilde{G}_l(y)}{\mathrm{d}y}, \frac{\mathrm{d}^2\tilde{G}_l(y)}{\mathrm{d}y^2}\right) + \mathrm{O}(n^{-2}) \tag{5.14}$$

where

$$Z(0, x, y, z) = 2^{d-\sigma}(2^{\sigma+2}x - 2y - 2^{\sigma} + 1)^{-1}\{(2x-y)^{-1}(4x-1)(3x-3y+\tfrac{1}{2}z) \\ -(4x-1)^{-1}[8(3-2^{\sigma})x^2 + (6.2^{\sigma} - 10 + 8y)x + 2y - 2^{\sigma} + 1]\}. \tag{5.15}$$

Substituting

$$\tilde{G}_l(y) = \tilde{G}^*(y) + \sum_{k=0}^{\infty} a_k^{(l)} y^k \tag{5.16}$$

into (5.14), where $\tilde{G}^*(y)$ is the fixed point of (5.14), with zeroth-order term given by (5.9) and (5.10), and linearising, we obtain

$$a_k^{(l+1)} = \sum_{k'=0}^{\infty} [V_{kk'}^{(0)} + n^{-1} V_{kk'}^{(1)} + O(n^{-2})] a_{k'}^{(l)} \tag{5.17}$$

where $\mathbf{V}^{(0)}$ is a diagonal matrix with elements

$$V_{kk'}^{(0)} = 2^{d-\sigma-\sigma k}\delta_{kk'}, \qquad k, k' = 0, 1, 2, \ldots. \tag{5.18}$$

Thus, the $1/n$ correction to the largest eigenvalue $\Lambda_1 = 2^{y_E}$ is simply $n^{-1}V_{0,0}^{(1)}$ which is in turn given by

$$n^{-1}V_{0,0}^{(1)} = n^{-1}\frac{\partial}{\partial \tilde{G}^*(y)} Z\left(y, \tilde{G}^*(y), \frac{d\tilde{G}^*(y)}{dy}, \frac{d^2\tilde{G}^*(y)}{dy^2}\right)\Bigg|_{y=0} \\ = n^{-1}\frac{\partial}{\partial x} Z(0, x, A_1^*, 2A_2^*)\Big|_{x=A_0^*} + O(n^{-2}) \tag{5.19}$$

where A_k^*, $k = 0, 1, 2$ are given by (5.10). Our final result for y_E is

$$y_E = d - \sigma + n^{-1}(\ln 2)^{-1}\Phi(2^d, 2^{\sigma}) + O(n^{-2}) \tag{5.20}$$

where

$$\Phi(r, s) = 2(r-s)^2(s^2-r)(r-1)(s^3+2s^2+2rs+r)[r(s-1)^2(r+s)^2(s^3-r)]^{-1}. \tag{5.21}$$

This result checks with the $\Delta\sigma$ expansion (3.11) to second order in $\Delta\sigma$ and to first order in n^{-1}. In figure 1, we plotted the approximate values of y_E for $n = 3$, $d = 1$ by including only the first-order term in n^{-1} in (5.20) to compare it with the actual values of y_E obtained numerically. We see that the approximate y_E obtained in this way is not at all accurate, especially near $\sigma = \frac{1}{2}$.

6. Summary and discussion

In this paper, we have obtained exact renormalisation group recursion relations for the n-vector and Heisenberg version of Dyson's HM in d dimensions and discussed their respective critical properties. We only considered the interaction which is rotationally invariant in spin space and in this case, the RGE are non-linear integral transformations which involve only one integration for all cases considered. It is interesting to note here the resemblance of our RGE to the approximate RG transformation for the short-range interaction n-vector model discussed by Kadanoff *et al* (1976). In fact, equation (80) of their work, which is the RG transformation for interactions in the one-hypercube approximation, can be rewritten in the form

$$\tilde{v}'([2^d p(1-p)]^{1/2} y)$$
$$= \text{constant} - \tfrac{1}{2} n \ln(1-p)$$
$$+\ln \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2} p 2^{-d} (\boldsymbol{x}-\boldsymbol{y})^2 + 2^d \tilde{v}(x)]\, \mathrm{d}\boldsymbol{x} \qquad (6.1)$$

where we have subtracted the Gaussian fixed point part from their $v(x)$; i.e. $\tilde{v}(x) = v(x) - \frac{1}{2} 2^{-2d} x^2$. For the meaning of $v(x)$, see equation (72) of Kadanoff *et al* (1976). Here, p is their variational parameter, $x = \|\boldsymbol{x}\|$, and $\boldsymbol{x}$ is an n-dimensional vector. This is to be compared with (2.6) or (3.5). Thus, the $\eta = 0$ approximation which amounts to putting $p = \frac{1}{2}$ in (6.1) for all d is equivalent to our n-vector HM with $\sigma = 2$.

For the spin-S Heisenberg HM, the renormalisation group transformation depends on both the spin magnitude S, and l, the number of times the transformation has been applied previously. However, since the critical exponents are determined by the transformation in the $l \to \infty$ limit which reduces to that of the $S = \infty$ classical Heisenberg HM for all S, we have an explicit form of *universality* in S for the quantum HM.

In order to assess quantum effects in the temperature range where experiments are usually performed, we have plotted in figure 2 the logarithm of the high-temperature susceptibilities against $\ln t$, where $t = 1 - \beta/\beta_c$ for $S = \frac{1}{2}$ and $S = \infty$ respectively for the particular case of $d = 3$ and $\sigma = 2$. The figure shows that the two curves almost parallel each other for a wide range of t, indicating the insignificance of quantum

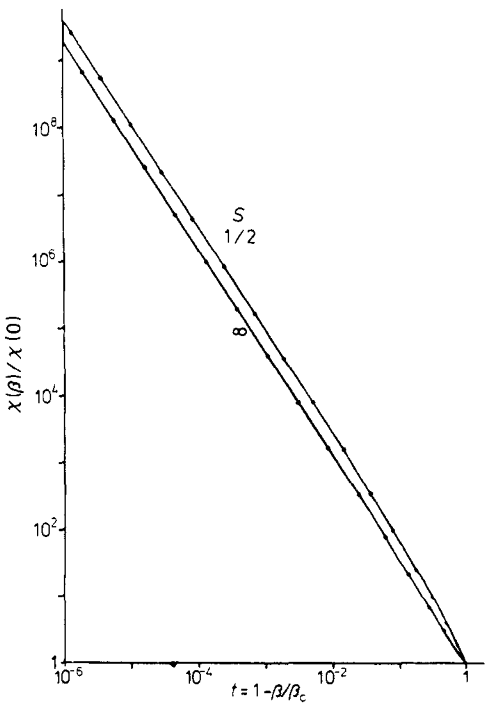

Figure 2. χ against $t = 1 - \beta_c/\beta$ in log–log plot for $S = \frac{1}{2}$ and $S = \infty$ Heisenberg HM respectively at $d = 3$, $\sigma = 2$. $\beta_c = 3{\cdot}111600073$ and $7{\cdot}122315720$ for $S = \frac{1}{2}$ and $S = \infty$ respectively and $\gamma = 1{\cdot}5247$ for both cases.

effects for this model. However, if one only used the data in the region $t \gtrsim 10^{-3}$, one might conclude that γ for spin-$\frac{1}{2}$ is larger than for the spin-infinity case. For the Heisenberg model with *short*-range interactions, the evidence for universality of critical exponents in S comes mainly from series analysis of the high-temperature susceptibility series, the conclusions of which are subject to personal opinion (Stanley 1974). Even though the interaction for the HM is not very physical, we may infer from our investigation that the quantum effects are indeed small in the critical region.

The d-dimensional generalisations considered here, following Blekher and Sinai (1974), are slightly different to those employed by Baker (1972) and Baker and Golner (1977). (The two are the same for $d = 1$ apart from insignificant surface terms.) When the critical exponent γ is considered as a function of σ/d and d, the $\Delta\sigma$ expansion obtained in § 3 shows that γ *does* depend on σ/d and d separately although the d dependence, which comes through $\Psi(d)$, (3.12), is rather weak. This weak d dependence is also seen from numerical calculations discussed in § 3. At $\sigma/d = 0{\cdot}65$ and $n = 1$ for example, the value of γ for $d = 3$ has changed only by 0·1% from that for $d = 1$ and the change for $d = 2$ is even smaller. If one were given only the numerical values of γ for $d = 1$ and $d = 2$ to four significant digits near $\sigma/d = \frac{2}{3}$, for example, one might be tempted to conclude that γ is only a function of σ/d. The $\Delta\sigma$ expansion also reveals that the n dependence of γ (or y_E) for HM is exactly the same as that for the power law interaction model obtained by Fisher *et al* (1972) to second order in $\Delta\sigma$. This suggests the close similarity of the two models near $\sigma = d/2$ so long as $\sigma < 2$. In any case, the HM is designed to simulate the power law interaction. However, the power law model assumes short-range behaviour whenever $\sigma > 2$ while the HM does not have such a property. The limiting behaviour of y_E as $\sigma \to 1$ for $d = 1$ was investigated for the power law case by Kosterlitz (1976). Our numerical results shown in figure 1, are consistent with his result for $n \geq 2$ in that $y_E \sim 1 - \sigma$ as $\sigma \to 1$. However the two models have different behaviour for $n = 1$. (The behaviour of y_E in the $\sigma \to 1$ limit for $d = n = 1$ was discussed in detail in I.)

In § 4, we solved the n-vector HM explicitly in the $n \to \infty$ limit by transforming the recursion relation for the magnetisation of finite systems to one which could be explicitly summed. Our solution, in the form of an equation of state, was then found to be exactly the same as that of the spherical model. It is interesting to note that the spherical constraint does not appear in our solution of the $n \to \infty$ HM, but is naturally embedded in the solution itself.

The line of approach employed in § 4 was used in § 5 to arrive at a simple RGE in the $n \to \infty$ limit, where the non-Gaussian fixed point and scaling exponents could be easily found. On the basis of this approach we were able to obtain correction terms of order n^{-1} to the RGE, and thus obtain the $1/n$ expansion of y_E to first order in n^{-1}. In this procedure, the assumption that the fixed point $\tilde{G}^*(y)$ of equation (5.8) is analytic at $y = 0$ is crucial. Without this assumption, the fixed point is not unique. To be specific, if $\tilde{G}^*(y)$ is a fixed point solution of equation (5.8) then $\tilde{G}^*(y) + Ay^{(d-\sigma)/\sigma}$ is also a fixed point with the constant A being arbitrary. To zeroth order, this does not affect the scaling exponent $y_E = d - \sigma$. However to obtain the $1/n$ correction to y_E, $\tilde{G}^*(y)$ being analytic at $y = 0$ to zeroth order was a necessity.

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References

Baker G A Jr 1972 *Phys. Rev.* B **5** 2622–33
Baker G A Jr and Golner G R 1977 *Phys. Rev.* B to be published
Blekher P M and Sinai Ya G 1974 *Sov. Phys.-JETP* **40** 195–7
Dyson F J 1969 *Commun. Math. Phys.* **12** 91–107
—— 1971 *Commun. Math. Phys.* **21** 269–83
Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* vol. 2 (New York: McGraw-Hill)
—— 1954 *Tables of Integral Transforms* vol. 2 (New York: McGraw-Hill)
Fisher M E, Ma S-K and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917–20
Guttmann A J, Kim D and Thompson C J 1977 *J. Phys. A: Math. Gen.* **10** L125–8
Kac M and Thompson C J 1971 *Physica Norv.* **5** 163–8
Kadanoff L P, Houghton A and Yalabik M C 1976 *J. Statist. Phys.* **14** 171–203
Kim D and Thompson C J 1977 *J. Phys. A.: Math. Gen.* **10** 1579–98
—— 1978 *J. Phys. A: Math. Gen.* **11** 375–84
Kosterlitz J M 1976 *Phys. Rev. Lett.* **37** 1577–80
McGuire J B 1973 *Commun. Math. Phys.* **32** 215–30
Stanley H E 1974 *Phase Transitions and Critical Phenomena* vol. 3, eds C Domb and M S Green (New York: Academic) chap. 7
Wegner F J 1976 *Phase Transitions and Critical Phenomena* vol. 6, eds C Domb and M S Green (New York: Academic) chap. 2